Proof: Scalar Projection, Vector Projection and Finding the Angle Between Two Vectors

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Definitions

1. The Vector Projection can be thought of as “projecting” a given vector A onto another vector B. So if you have a given vector $A = (7, 7)$ and $B = (15, 0)$ the result, $proj(A, B) = (7, 0)$.
2. The Scalar Projection can be thought of as the length or magnitude of the projection of a given vector A onto another vector B. So if you have a given vector $A = (7, 7)$ and $B = (15, 0)$ the result $scalarProjection(A, B) = 7$, which you can plainly see is also the magnitude or length of $proj(A, B)$.
3. The angle between two vectors is pretty self-explanatory, but in this case we’ll only talk about 2D until after the proof.

Relative Angle Orientation & Scalar Projection

The first thing to understand is that the relative angle between the two vectors (θ) is the only angle that matters for projections. This could be calculated if each vector had a known angle (θ_A , θ_B) and taking the difference of them, $\theta = \theta_A - \theta_B$ but this does not easily extrapolate into higher dimensions beyond 2D when you don’t know the angles already. However a helpful understanding that can be grasped visually is the following,

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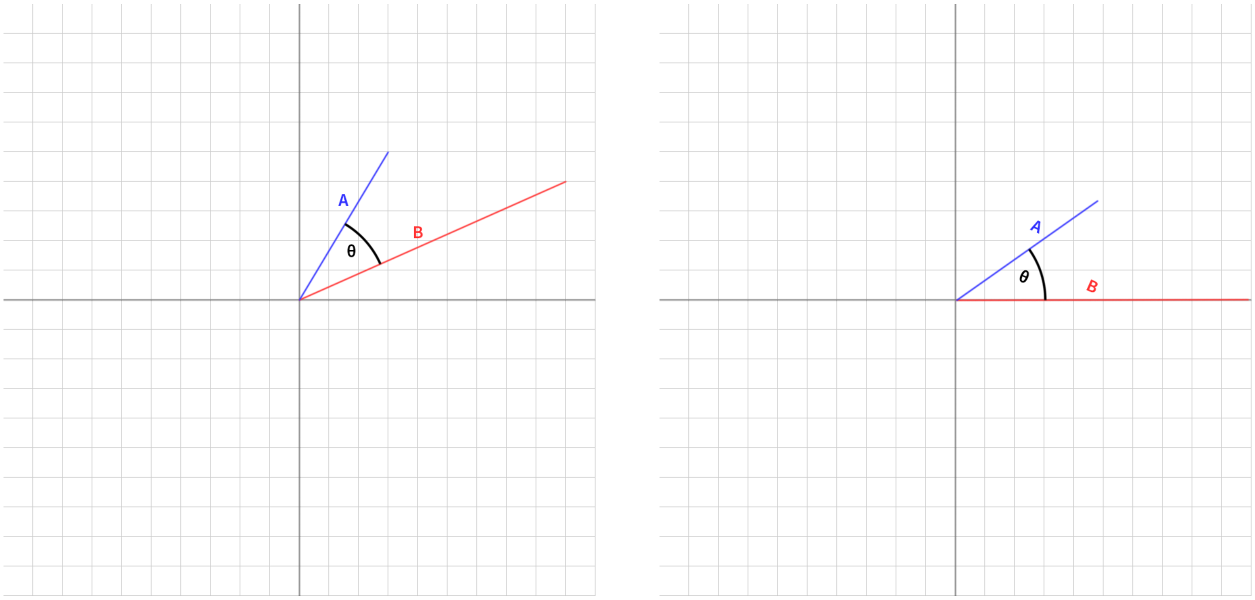

This is to say that you can rotate both angles so that B is flat on the x-axis for easier comprehension of the system. In these circumstances we can also benefit from the polar-to-Cartesian transform of the x-axis component of a vector.

$$x = r * cos(\theta)$$
$$r = \frac{x}{cos(\theta)}$$
$$\theta = cos^{-1}(\frac{x}{r})$$

This tells us that we can get the x-axis length of A by multiplying the length of A (r here) by the cosine of the angle between A and B (θ). We can calculate the value of r with the following equation,

$$r = \sqrt{x^2 + y^2}$$
$$||A|| = \sqrt{A_x^2 + A_y^2}$$

These two formula are the exact same thing, just written in two ways. From now on we'll refer to the length of A as $||A||$ instead of r. So the formula becomes,

$$x = ||A|| * cos(\theta)$$

Which is to say that x is the length of A whose component is projected onto B in the same way we treat normal vectors as breakdowns of components projected on the X and Y axis. This *is* the scalar product ($scalarProjection(A, B)$).

Interestingly enough to understand this another way we can use a different equation. If you recall from trigonometry that $cos(\theta)$ is the x-axis part of a vector whose length is 1, and that you can reduce a vector to a “unit vector” (a vector with a length of 1) via the formula,

$$\hat{B} = (\frac{B_x}{B_x + B_y}, \frac{B_y}{B_x + B_y})$$

Then you can figure out that

$$\frac{B_x}{B_x + B_y} = cos(\theta)$$

Given this you can alternatively calculate the scalar projection of (A,B) via the formula,

$$A \cdot \hat{B} = A_x \hat{B}x + A_y \hat{B}_y$$

Where the · symbol means “dot product”, which is defined as,

$$A \cdot B = \sum_{i=1}^{n} A_i B_i$$

for n-dimensions. This means the above formula ($A \cdot \hat{B}$) turns out to the following,

$$A \cdot \hat{B} = A_x \hat{B}_x + A_y 0 = A_x \hat{B}_x$$

Due to the relative rotation of B acting as the x-axis and therefore not having a relative Y component. Just imagine it as the version of the two angles in the second figure above - with B laying flat on the X axis.

This means that you can equate the scalarProjection(A,B) (which we’ll just write as S(A,B)) to the two formula:

$$S(A, B) = A \cdot \hat{B} = ||A||cos(\theta)$$

Calculating Cosine

Yet, how do we calculate cosine if we only have A and B and are unable to rotate them like in the above graphical representation? Well with some algebraic rearrangement we can deduce a formula from the definition of the scalar projection.

We know that,

$$A \cdot \hat{B} = ||A||cos(\theta)$$

Further we can define the relationship between B and $\hat{B}$ via the formula,

$$B = \hat{B} * ||B||$$
$$\hat{B} = \frac{B}{||B||}$$

And because dot-products have the commutative multiplication property,

$$A \cdot Bc = Ac \cdot B = c(A \cdot B)$$

We can deduce,

$$A \cdot B * ||B||^{-1} = \frac{A \cdot B}{||B||}$$

Therefore,

$$\frac{A \cdot B}{||B||} = ||A||cos(\theta)$$

Which we can simplify into,

$$\frac{A \cdot B}{||A|| * ||B||} = cos(\theta)$$

and derive the value of θ via,

$$\theta = cos^{-1}(\frac{A \cdot B}{||A|| * ||B||})$$

Which is the angle between the two vectors.

Vector Projection

Now, given that we know the angle between two vectors (θ) and the scalar project ($s = S(A, B)$) we can now calculate the vector projection($v = V(A, B)$) very easily.

Recall the vector projection is a projection of the components of A onto B. All we have to do to do this is get the normalized vector of B ($\hat{B}$) and multiply it by the length of A projected onto B - aka the Scalar Product (s),

$$v = s\hat{B} = ||A||cos\theta * \frac{B}{||B||}$$

Now that might seem like a rather poor example, but what we can consider it as is that we are taking the vector length of A projected on B (s) and then we're multiplying it by a unit-vector representing the direction of B ($\hat{B}$) which gives us a vector in the direction of B with the length of the A vector components when projected onto B.

If that is confusing, an example will likely clear things up far better than words!

Example 1

Given two vectors,

$$A = (7, 7)$$
$$B = (15, 0)$$
$$\hat{B} = (1, 0)$$

The scalar projection is,

$$s = A \cdot \hat{B} = (7 * 1) + (7 * 0) = 7$$

Which - keep in mind is not the distance of A but the length of A projected onto B. Now given that, we can find the angle between the two,

$$\theta = cos^1(\frac{A \cdot B}{||A|| * ||B||})$$
$$A \cdot B = (7 * 15) + (7 * 0) = 105$$
$$||A|| = \sqrt{7^2 + 7^2} = \sqrt{98} = 9.8995$$
$$||B|| = \sqrt{15^2 + 0^2} = \sqrt{15^2} = 15$$
$$||A|| * ||B|| = 148.4925$$
$$\frac{A \cdot B}{||A|| * ||B||} = \frac{105}{148.4925} = 0.707$$
$$cos^1(0.707) = 0.7856 \ or \ 45.01^{\circ}$$

Which makes sense - A is rising and extending at the same amount and B is an x-axis only vector. So there's a 45* angle between them, with the 0.01 being the margin of error from rounding the various roots in the formula.

Given this, we can then perform the vector projection,

$$v = s\hat{B} = 7 * (1, 0) = (7, 0)$$

So we now can see - we projected the components of A onto B, yielding a vector in the direction of B ($(1, 0)$) but with length of the A component (7), yielding the vector ($(7, 0)$).

Hopefully that makes some sense.

Conclusion

This has been a review of the manner I worked through in order to understand how Scalar Projection and Vector Projection work, as well as how to calculate the angle between two vectors. A useful bonus - and why this method is better than simple trigonometry is that it *scales to any number of dimensions*. Given we can define the magnitude operations (ex: $||A||$) and the dot product on two vectors, we can calculate the angle between them, and perform all manner of projection on them.

Thus I felt - should I ever forget this, or need to explain to another person this would be a worthwhile explanation and/or proof.
